

## ON DIV-CURL FOR HIGHER ORDER

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*For Eli*

ABSTRACT. We present new examples of complexes of differential operators of order  $k$  (any given positive integer) that satisfy div-curl and/or  $L^1$ -duality estimates.

## 1. INTRODUCTION

In 2004 Stein and the first named author discovered a connection [LS] between the celebrated Gagliardo-Nirenberg inequality [G]-[N] for functions

$$(1) \quad \|f\|_{L^r(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^1(\mathbb{R}^n)}, \quad r = n/(n-1)$$

and a recent estimate of Bourgain and Brezis [BB2] for divergence-free vector fields as proved by Van Schaftingen [VS1]

$$(2) \quad \|Z\|_{L^r(\mathbb{R}^n)} \leq C \|\operatorname{Curl} Z\|_{L^1(\mathbb{R}^n)}, \quad r = n/(n-1), \quad \operatorname{div} Z = 0$$

Such connection is provided by the exterior derivative operator acting on differential forms on  $\mathbb{R}^n$  with (say) smooth and compactly supported coefficients

$$(3) \quad d : \Lambda_q(\mathbb{R}^n) \rightarrow \Lambda_{q+1}(\mathbb{R}^n), \quad 0 \leq q \leq n$$

It was proved in [LS] that the inequality

$$(4) \quad \|u\|_{L^r(\mathbb{R}^n)} \leq C(\|du\|_{L^1(\mathbb{R}^n)} + \|d^*u\|_{L^1(\mathbb{R}^n)}), \quad r = n/(n-1)$$

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holds for any form  $u$  of degree  $q$  other than  $q = 1$  (unless  $d^*u = 0$ ) and  $q = n - 1$  (unless  $du = 0$ ). Note that (1) is the case  $q = 0$ , whereas (2) is the case  $q = 1$  specialized to  $d^*u = 0$ .

Since those earlier results div/curl-type phenomena have been studied both in the Euclidean and non-Euclidean settings [Am], [BV], [HP1], [HP2], [M], [MM], [Mi] [VS4], [CV], [Y]. In [VS2] and the recent works [BB3], [VS3], [VS5], differential conditions of higher order have been considered for the first time in such context. (By contrast, the exterior derivative in (3) is defined in terms of differential conditions of order 1.)

The goal of the present paper is to produce a new class of differential operators of order  $k$  (where  $k$  is any given positive integer) that satisfy an appropriate analogue of (4) and contain the operators introduced in [BB3], [VS2] and [VS3]; since the conditions

$$(5) \quad d \circ d = 0; \quad d^* \circ d^* = 0$$

play an important role in the proof of (4), the new operators should satisfy (5) as well. We achieve this goal in a number of ways, beginning with:

**Theorem 1.1.** *If  $u \in C_q^\infty(\mathbb{R}^n)$  has compact support, then*

$$(6) \quad \|u\|_{W^{k-1,r}} \leq C(\|\mathcal{T}u\|_{L^1} + \|\mathcal{T}^*u\|_{L^1}), \quad r = n/(n-1)$$

*whenever  $q$  is neither 1 (unless  $\mathcal{T}^*u = 0$ ) nor  $n - 1$  (unless  $\mathcal{T}u = 0$ ), where*

$$(7) \quad \mathcal{T}u := \sum_{|L|=q+1} \left( \sum_{\substack{|I|=q \\ j=1,\dots,n}} \epsilon_L^{jI} \frac{\partial^k u_I}{\partial x_j^k} \right) dx^L$$

Here and in the sequel,  $W^{a,p}(\mathbb{R}^n)$  denotes the Sobolev space consisting of  $a$ -times differentiable functions in the Lebesgue class  $L^p(\mathbb{R}^n)$  (and  $W_q^{a,p}(\mathbb{R}^n)$  will denote the space of  $q$ -forms with coefficients in  $W^{a,p}(\mathbb{R}^n)$ ), while  $\epsilon_C^{AB} \in \{-1, 0, +1\}$  is the sign of the permutation that carries the ordered set  $AB = \{a_1, \dots, a_\ell, b_1, \dots, b_q\}$  to the label  $C = (c_1, \dots, c_{\ell+q})$ , if these have identical content, and is otherwise zero. Note that when  $k = 1$  then  $\mathcal{T} = d$  and inequality (6) is indeed (4).

Another such complex, again involving a differential condition of order  $k \geq 1$ , is obtained by embedding  $\mathbb{R}^n$  isometrically in a larger space  $\mathbb{R}^N$ . (The choice of “inflated” dimension  $N$  will be discussed later.) The resulting operators act on “hybrid  $\mathbb{R}^n$ -to- $\mathbb{R}^N$ ” spaces of forms whose coefficients are trivial extensions to  $\mathbb{R}^N$  of functions defined on  $\mathbb{R}^n$ ; to distinguish such spaces from the classical Sobolev spaces

$W_q^{a,p}(\mathbb{R}^N)$  (to which they are by necessity transversal) we will use the notation

$$\widetilde{W}_q^{a,p}(\mathbb{R}^N), \quad 0 \leq q \leq N,$$

and we will write

$$\widetilde{C}_q^{\infty,c}(\mathbb{R}^N), \quad 0 \leq q \leq N$$

to indicate a dense subspace of smooth “compactly supported” forms. These operators, which we denote  $\widetilde{T}_{1,\aleph}$ , map

$$\widetilde{T}_{1,\aleph} : \widetilde{C}_q^{\infty,c}(\mathbb{R}^N) \rightarrow \widetilde{C}_{q+1}^{\infty,c}(\mathbb{R}^N), \quad 0 \leq q \leq N$$

The label  $\aleph$  refers to a choice of an ordering for the set of all  $k$ -th order derivatives in  $\mathbb{R}^n$ , and so in practice we define a finite family  $\{\widetilde{T}_{1,\aleph}\}_{\aleph}$  of such complexes. (We use the subscript “1” in  $\widetilde{T}_{1,\aleph}$  to specify that  $\widetilde{T}_{1,\aleph}$  maps  $q$ -forms to  $(q+1)$ -forms, a distinction that will be needed later on). The explicit definition of  $\widetilde{T}_{1,\aleph}$  will be given in the next section; what matters here is that these operators satisfy a more general version of (6) in the sense that the following inequality implies (6) but the converse is not true:

**Theorem 1.2.** *If  $U \in \widetilde{C}_q^{\infty}(\mathbb{R}^N)$  has compact support, then*

$$(8) \quad \|U\|_{\widetilde{W}^{k-1,r}} \leq C(\|\widetilde{T}_{1,\aleph}U\|_{\widetilde{L}^1} + \|\widetilde{T}_{1,\aleph}^*U\|_{\widetilde{L}^1}), \quad r = n/(n-1)$$

*whenever  $q$  is neither 1 (unless  $\widetilde{T}_{1,\aleph}^*U = 0$ ) nor  $N-1$  (unless  $\widetilde{T}_{1,\aleph}U = 0$ ).*

Theorem 1.1 recaptures an  $L^1$ -duality estimate of Bourgain and Brezis:

**Theorem 1.3** ([BB3]). *Let  $k \geq 1$ . For every vector field  $u \in L^1(\mathbb{R}^n; \mathbb{R}^n)$  if*

$$(9) \quad \sum_{j=1}^n \frac{\partial^k u_j}{\partial x_j^k} = 0$$

*in the sense of distributions, then*

$$(10) \quad \left| \int_{\mathbb{R}^n} u_j \cdot h_j \right| \leq C \|u\|_{L^1} \|\nabla h\|_{L^n}, \quad j = 1, \dots, n$$

*for any  $h \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n; \mathbb{R}^n)$ , where the constant  $C$  only depends on the dimension of the space  $n$  and on the order  $k$ .*

On the other hand, Theorem 1.2 was motivated by a recent result of van Schaftingen:

**Theorem 1.4** ([VS3]). *Given  $k \geq 1$  and  $n \geq 2$ , let*

$$(11) \quad m := \binom{n-1+k}{k}$$

*For any vector field  $g = (g_\alpha)_{\alpha \in \mathcal{S}(n,k)} \in L^1(\mathbb{R}^n; \mathbb{R}^m)$  if*

$$(12) \quad \sum_{\alpha \in \mathcal{S}(n,k)} \frac{\partial^k g_\alpha}{\partial x^\alpha} = 0$$

*in the sense of distributions, then*

$$(13) \quad \left| \int_{\mathbb{R}^n} g_\alpha \cdot h_\alpha \right| \leq C \|g\|_{L^1} \|\nabla h\|_{L^n}, \quad \alpha \in \mathcal{S}(n,k)$$

*for any  $h \in (W^{1,n} \cap L^\infty)(\mathbb{R}^n; \mathbb{R}^m)$ , where the constant  $C$  only depends on the dimension of the space  $n$  and on the order  $k$ .*

Here  $\mathcal{S}(n,k)$  denotes the set of  $k$ -multi-indices in  $\mathbb{R}^n$ :

$$(14) \quad \mathcal{S}(n,k) = \left\{ \alpha = (\alpha_1, \dots, \alpha_n) \left| \alpha_t \in \{0, 1, \dots, k\}, \sum_{t=1}^n \alpha_t = k \right. \right\}.$$

A key ingredient in the proof of Theorems 1.1 and 1.2 is the fact that the Hodge laplacians for these operators, namely

$$\square_{\mathcal{T}} := \mathcal{T}\mathcal{T}^* + \mathcal{T}^*\mathcal{T} : C_q^{\infty,c}(\mathbb{R}^n) \rightarrow C_q^{\infty,c}(\mathbb{R}^n), \quad 0 \leq q \leq n$$

and

$$\tilde{\square}_{1,\mathbb{N}} := \tilde{T}_{1,\mathbb{N}} \tilde{T}_{1,\mathbb{N}}^* + \tilde{T}_{1,\mathbb{N}}^* \tilde{T}_{1,\mathbb{N}} : \tilde{C}_q^{\infty,c}(\mathbb{R}^N) \rightarrow \tilde{C}_q^{\infty,c}(\mathbb{R}^N), \quad 0 \leq q \leq N$$

satisfy a uniform Legendre-Hadamard condition which in turn yields elliptic estimates.

Rather surprisingly, it turns out that in fact there is a larger class of such operators, mapping

$$\tilde{T}_{\ell,\mathbb{N}} : \tilde{C}_q^{\infty,c}(\mathbb{R}^N) \rightarrow \tilde{C}_{q+\ell}^{\infty,c}(\mathbb{R}^N), \quad 0 \leq q \leq N$$

where the label  $\ell$  now runs over all the elements of what we call *the set of admissible degree increments*, which is a subset of  $\{1, \dots, k\}$  determined by  $n$  (the dimension of the source space) and  $k$  (the order of differentiation): for any  $n \geq 2$  and  $k \geq 2$ , the set of admissible degree increments contains at least two distinct elements:  $\ell = 1$  (discussed earlier) and also  $\ell = k$ . Each admissible degree increment in turn determines an “inflated dimension”  $N$  (in particular  $N$  will change with  $\ell$ ). However the situation for  $\ell \neq 1$  differs from the case  $\ell = 1$  in two important respects: the crucial condition (5) will hold only for odd  $\ell$ , and if  $\ell \neq 1$  the Hodge laplacian for  $\tilde{T}_{\ell,\mathbb{N}}$  will fail to be

uniformly elliptic (even for  $\ell$  odd): as a result there is no analog of (6). Instead, we show that for any admissible degree increment (thus also for  $\ell = 1$ ), the operators  $\tilde{T}_{\ell, \aleph}$  satisfy  $L^1$ -duality estimates that are similar in spirit, and indeed are equivalent to (13); see Theorem 2.3 for the precise statement.

A further class of operators which contains our very first example  $\mathcal{T}$ , see (7), can be defined in terms of  $\tilde{T}_{\ell, \aleph}$  and of the aforementioned embedding:  $\mathbb{R}^n \rightarrow \mathbb{R}^N$ . Such operators map

$$\mathcal{T}_{\ell, \aleph} : C_q^{\infty, c}(\mathbb{R}^n) \rightarrow C_{q+\ell}^{\infty, c}(\mathbb{R}^n)$$

and satisfy div-curl and/or  $L^1$ -duality estimates that are stated solely in terms of the source space  $\mathbb{R}^n$  rather than the “hybrid  $\mathbb{R}^n$ -to- $\mathbb{R}^N$ ” spaces  $\tilde{L}_q^p(\mathbb{R}^N)$  and  $\tilde{W}_q^{a, p}(\mathbb{R}^N)$ , see Theorem 2.4 and (77).

(Of course, if  $\ell \neq 1$  such operators are non-trivial only for  $n \geq \ell$ .)

We need to explain the reason for our choice to keep track, through the label  $\aleph$ , of the orderings of  $\mathcal{S}(n, k)$ : this has to do with the notion of invariance. One would like to know whether the identity

$$(15) \quad \tilde{T}_{\ell, \aleph} \Psi^* F = \Psi^* \tilde{T}_{\ell, \aleph} F$$

holds for any  $F \in \tilde{C}_q^\infty(\mathbb{R}^N)$  and for some non-trivial class of diffeomorphisms

$$\Psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

of class  $C^{k+1}$ : it is in this context that the choice of  $\aleph$  may be relevant. In the case  $k = 1$  our construction gives  $N = n$  with  $\aleph$  spanning the set of all permutations of  $\{1, \dots, n\}$ , and since  $k$  is 1 there is only the admissible degree increment  $\ell = 1$ . As a result, for  $k = 1$  we have

$$\tilde{T}_{1, \aleph} U = \sum_{|L|=q+1} \left( \sum_{\substack{|I|=q \\ j=1, \dots, n}} \epsilon_L^{\aleph(j)I} \frac{\partial U_I}{\partial x_j} \right) dx^L, \quad \aleph \in \Sigma(1, \dots, n).$$

In particular one has

$$\tilde{T}_{1, \aleph_0} = \mathcal{T} = d$$

for exactly one permutation  $\aleph_0$  (the identity) which therefore determines an invariant operator. On the other hand it is easy to check that for any  $\aleph \neq \aleph_0$  the operators  $\tilde{T}_{1, \aleph}$  fail to be invariant.

No such phenomenon exists for  $k \geq 2$ : there is no choice of  $\aleph$  (nor  $\ell$ ) that makes  $\tilde{T}_{\ell, \aleph}$  invariant and (15) fails even in the case when  $\Psi$  originates from a rotation of  $\mathbb{R}^n$ . It can be verified that  $\mathcal{T}_{\ell, \aleph}$ , too, is not invariant because if  $k \geq 2$  the identity

$$(16) \quad \mathcal{T}_{\ell, \aleph} \psi^* = \psi^* \mathcal{T}_{\ell, \aleph}$$

fails for any  $\ell$  and for any  $\aleph$ , already for  $\psi$  a rotation of  $\mathbb{R}^n$ .

Finally, we point out that our results can be rephrased in terms of the canceling and cocanceling conditions of [VS4]: within that framework, our results provide new classes of differential operators of *arbitrary* order that are canceling and/or cocanceling, with the size of the admissible degree increments acting as an indicator of the canceling property. See the remarks in Section 4.

This paper is organized as follows: in Section 2 we introduce the notion of admissible degree increment, we describe the “hybrid  $\mathbb{R}^n$ -to- $\mathbb{R}^N$ ” Sobolev spaces  $\widetilde{W}_q^{a,p}(\mathbb{R}^N)$  in term of the embedding, and we define the operators  $\widetilde{T}_{\ell,\aleph}$  and  $\mathcal{T}_{\ell,\aleph}$  and discuss their basic properties (adjoints; uniform ellipticity). The  $L^1$ -duality estimates for  $\widetilde{T}_{\ell,\aleph}$  and for  $\mathcal{T}_{\ell,\aleph}$  are stated in Theorems 2.3 and 2.4, and the precise statements of (8) and of (6) are given in Theorem 2.8 and in (77). All the proofs are deferred to Section 3. Section 4 contains some remarks and a few questions.

**1.1. Notation.** As customary, we let  $\Lambda_q(\mathbb{R}^n)$  denote the space of  $q$ -forms:

$$(17) \quad \Lambda_q(\mathbb{R}^n) = \left\{ f = \sum_{I \in \mathcal{I}(n,q)} f_I dx^I \mid f_I : \mathbb{R}^n \rightarrow \mathbb{R} \right\}, \quad 0 \leq q \leq n$$

where  $\mathcal{I}(n, q)$  denotes the set of  $q$ -labels for  $\mathbb{R}^n$ :

$$(18) \quad \mathcal{I}(n, q) = \{ I = (i_1, \dots, i_q) \mid i_t \in \{1, \dots, n\}, i_t < i_{t+1} \}$$

and

$$dx^I = dx_{i_1} \wedge \dots \wedge dx_{i_q}.$$

When  $q = n$  the expression above is the volume form and we use the notation  $dV$ . We will regard the label set  $\mathcal{I}(n, q)$  as canonically ordered (alphabetical ordering). Letting

$$i : \mathbb{R}^n \rightarrow \mathbb{R}^N$$

denote the isometric embedding mentioned above and defined in (26), the “hybrid  $\mathbb{R}^n$ -to- $\mathbb{R}^N$ ” subspace of  $\Lambda_q(\mathbb{R}^N)$  (consisting of those  $q$ -forms whose coefficients are trivial extensions to  $\mathbb{R}^N$  of functions defined on  $\mathbb{R}^n$ ) is more precisely described as follows

$$(19) \quad \widetilde{\Lambda}_q(\mathbb{R}^N) := \left\{ F = \sum_{I \in \mathcal{I}(N,q)} F_I dz^I \mid F_I \circ i \circ \pi = F_I \right\}, \quad 0 \leq q \leq N$$

where

$$\pi : \mathbb{R}^N \rightarrow \mathbb{R}^n$$

is chosen so that

$$(20) \quad (\pi \circ i)(x) = x, \quad \text{for all } x \in \mathbb{R}^n$$

As a result the reverse composition

$$(21) \quad i \circ \pi : \mathbb{R}^N \rightarrow \mathbb{R}^N$$

is a projection.

We will denote the Hodge-star operators for each of  $\Lambda_q(\mathbb{R}^n)$  and  $\Lambda_q(\mathbb{R}^N)$  respectively by  $*_n$  and  $*_N$ ; note that we have

$$(22) \quad *_N : \tilde{\Lambda}_q(\mathbb{R}^N) \rightarrow \tilde{\Lambda}_{N-q}(\mathbb{R}^N), \quad 0 \leq q \leq N.$$

## 2. STATEMENTS

**2.1. Admissible degree increments.** Given three integers:

- i.  $n \geq 2$  (the dimension of the source space),
- ii.  $k \geq 1$  (the order of the differential condition), and
- iii.  $1 \leq \ell \leq k$ ,

we say that  $\ell$  is an *admissible degree increment for the pair  $(n, k)$*  if and only if the polynomial equation

$$(23) \quad \binom{N}{\ell} = \binom{n-1+k}{k}$$

has a solution  $N$  that satisfies the following two conditions:

$$(24) \quad N \in \mathbb{Z}^+, \quad N \geq n-1+\ell.$$

Note that the pair  $(n, 1)$  (that is,  $k = 1$ ) has *exactly one* admissible degree increment, namely  $\ell = 1$ , and in this case equation (23) has the unique solution:  $N = n$ . On the other hand, for  $k \geq 2$  any pair  $(n, k)$  will have *at least two* admissible degree increments ( $\ell = 1, k$ ) and possibly more, for instance: the pair  $(n, k) = (2, 9)$  has (exactly) four admissible degree increments, namely  $\ell = 1, 2, 3, 9$ ; similarly, the pair  $(n, k) := (2, 29)$  has (at least)  $\ell = 1, 2, 29$ . For any admissible degree increment, we consider the embedding

$$(25) \quad i : \mathbb{R}^n \rightarrow \mathbb{R}^N$$

defined as follows

$$(26) \quad i(x_1, \dots, x_n) = (z_1, \dots, z_N) := (x_1, \dots, x_n, 0, \dots, 0)$$

where  $N = N(n, k, \ell)$  is as in (23) and (24). We let  $i$  also denote the embedding of  $k$ -multi-indices

$$i : \mathcal{S}(n, k) \rightarrow \mathcal{S}(N, k)$$

that is canonically induced by (26), namely

$$(27) \quad i(\alpha_1, \dots, \alpha_n) := (\alpha_1, \dots, \alpha_n, 0, \dots, 0) \in \mathcal{S}(N, k)$$

and adopt the notation

$$(28) \quad i\mathcal{S}(n, k) := \{i\alpha \mid \alpha \in \mathcal{S}(n, k)\} \subsetneq \mathcal{S}(N, k)$$

We have

$$(29) \quad |i\mathcal{S}(n, k)| = m$$

with  $m = m(n, k)$  as in (11), and so there are  $m!$ -many distinct orderings of  $i\mathcal{S}(n, k)$ . By the definition of  $N$  the set of labels  $\mathcal{I}(N, \ell)$  also has cardinality  $m$  and we will think of each ordering of  $i\mathcal{S}(n, k)$  as a one-to-one correspondence

$$(30) \quad \aleph : i\mathcal{S}(n, k) \rightarrow \mathcal{I}(N, \ell); \quad \aleph^{-1} : \mathcal{I}(N, \ell) \rightarrow i\mathcal{S}(n, k)$$

**2.2. Hybrid Function spaces.** Given an integer  $a \geq 0$  and given  $p, p' \geq 1$  such that  $1/p + 1/p' = 1$ , we first set ( $q = 0$ )

$$(31) \quad \tilde{L}^p(\mathbb{R}^N) := \{F : \mathbb{R}^N \rightarrow \mathbb{R} \mid F \circ i \circ \pi = F, \quad F \circ i \in L^p(\mathbb{R}^n, dV)\}$$

where  $i$  is as in (26) and

$$(32) \quad \pi(z_1, \dots, z_n, \dots, z_N) = (x, \dots, x_n) := (z_1, \dots, z_n)$$

satisfies (20), and

$$\widetilde{W}^{a,p}(\mathbb{R}^N) := \left\{ F : \mathbb{R}^N \rightarrow \mathbb{R} \mid \partial^\lambda F \in \tilde{L}^p(\mathbb{R}^N), \lambda \in \mathcal{S}(N, s), 0 \leq s \leq a \right\}.$$

Note that if  $F \in \widetilde{W}^{a,p}(\mathbb{R}^N)$  then it follows from

$$F \circ i \circ \pi = F$$

that

$$(33) \quad \frac{\partial F}{\partial z_t} = 0, \quad \text{for any } t = n+1, \dots, N,$$

which in turn grants

$$(34) \quad \frac{\partial^s F}{\partial z^\lambda} = 0$$



for any  $1 \leq s \leq a$  and for any  $\lambda \in \mathcal{S}(N, s) \setminus i\mathcal{S}(n, s)$ , so that these spaces are more precisely described as follows

$$\widetilde{W}^{a,p}(\mathbb{R}^N) = \left\{ F : \mathbb{R}^N \rightarrow \mathbb{R} \mid \partial^{i\beta} F \in \widetilde{L}^p(\mathbb{R}^N), \beta \in \mathcal{S}(n, s), 0 \leq s \leq a \right\}.$$

As customary, these definitions are extended to forms  $F \in \widetilde{L}_q^p(\mathbb{R}^N)$  of any degree  $0 \leq q \leq N$  (resp.  $F \in \widetilde{W}_q^{a,p}(\mathbb{R}^N)$ ,  $0 \leq q \leq N$ ) by requiring that

$$F = \sum_{I \in \mathcal{I}(N,q)} F_I dz^I \quad \text{has} \quad F_I \in \widetilde{L}^p(\mathbb{R}^N) \quad (\text{resp. } F_I \in \widetilde{W}^{a,p}(\mathbb{R}^N))$$

for any  $I \in \mathcal{I}(N, q)$ . We observe for future reference that identity (20) grants

$$(35) \quad \left( \frac{\partial^s F}{\partial z^{i\beta}} \right) \circ i = \frac{\partial^s (F \circ i)}{\partial x^\beta}$$

for any  $\beta \in \mathcal{S}(n, s)$  and for any  $F \in \widetilde{W}^{a,p}(\mathbb{R}^N)$ .

**Lemma 2.1.** *For any  $0 \leq q \leq N$ ; for any  $p \geq 1$  and for any integer  $a \geq 1$  the following properties hold:*

i.  $\widetilde{L}_q^2(\mathbb{R}^N)$  is a Hilbert space with inner product

$$(36) \quad \langle F, G \rangle_{\widetilde{L}} := \int_{\mathbb{R}^n} *_n i^* *_N (F \wedge *_N G)$$

ii.  $\widetilde{L}_q^p(\mathbb{R}^N)$  is a Banach space with norm

$$(37) \quad \|F\|_{\widetilde{L}_q^p} := \left( \int_{\mathbb{R}^n} *_n i^* (*_N (F \wedge *_N F))^{p/2} \right)^{1/p}$$

iii.  $\widetilde{W}_q^{a,2}(\mathbb{R}^N)$  is a Hilbert space with inner product

$$(38) \quad (F, G)_{\widetilde{W}} := \sum_{\substack{0 \leq s \leq a \\ \beta \in \mathcal{S}(n,s)}} \langle D^{i\beta} F, D^{i\beta} G \rangle_{\widetilde{L}}$$

where we have set

$$(39) \quad D^{i\beta} F := \sum_{I \in \mathcal{I}(N,q)} (\partial^{i\beta} F_I) dz^I$$

iv.  $\widetilde{W}_q^{a,p}(\mathbb{R}^N)$  is a Banach space with norm

$$(40) \quad \|H\|_{\widetilde{W}_q^{a,p}} := \left( \sum_{\substack{I \in \mathcal{I}(N,q) \\ \beta \in \mathcal{S}(n,s) \\ 0 \leq s \leq a}} \|\partial^{i\beta} H_I\|_{L^p}^p \right)^{1/p}$$

v. The set

$$(41) \quad \widetilde{C}_q^{\infty,c}(\mathbb{R}^N) := \left\{ F = \sum_{I \in \mathcal{I}(N,q)} F_I dz^I \in \widetilde{\Lambda}_q(\mathbb{R}^N) \mid F_I \circ i \in C_c^\infty(\mathbb{R}^n) \right\}$$

is dense in  $\widetilde{L}_q^p(\mathbb{R}^N)$  (resp.  $\widetilde{W}_q^{a,p}(\mathbb{R}^N)$ ) with respect to the norm (37) (resp. (40)).

A sequence  $\{\Phi_j\}_j \subset \widetilde{C}_q^{\infty,c}(\mathbb{R}^N)$  is said to converge in the sense of the space  $\widetilde{\mathcal{D}}_q(\mathbb{R}^N)$  to  $\Phi \in \widetilde{C}_q^{\infty,c}(\mathbb{R}^N)$ , see [A], provided the following conditions are satisfied:

i. There is a set  $K \Subset \mathbb{R}^n$  such that

$$\text{Supp}((\Phi_j - \Phi) \circ i) \subset K \text{ for each } j$$

ii. For any  $0 \leq s < \infty$  and for each  $\beta \in \mathcal{S}(n, s)$  we have

$$\lim_{j \rightarrow \infty} \frac{\partial^\beta((\Phi_j)_I \circ i)}{\partial x^\beta} = \frac{\partial^\beta(\Phi_I \circ i)}{\partial x^\beta} \quad \text{uniformly in } K$$

for any  $I \in \mathcal{I}(N, q)$ .

There exists a locally convex topology on the vector space  $\widetilde{C}_q^{\infty,c}(\mathbb{R}^N)$  with respect to which a linear functional  $\mathcal{L}$  is continuous if, and only if,  $\mathcal{L}(\Phi_j) \rightarrow \mathcal{L}(\Phi)$  whenever  $\Phi_j \rightarrow \Phi$  in the sense of the space  $\widetilde{\mathcal{D}}_q(\mathbb{R}^N)$ .

For any  $1 \leq p, p' < \infty$  with  $1/p + 1/p' = 1$ , the dual space  $\widetilde{W}_q^{-a,p'}(\mathbb{R}^N)$  of  $\widetilde{W}_q^{a,p}(\mathbb{R}^N)$  is identified (in the usual fashion, see e.g., [A, III.3.8 – III.3.12]) with a closed subspace of the Cartesian product

$$(\widetilde{L}^{p'}(\mathbb{R}^N))^{\binom{N}{q}M} \quad \text{where} \quad M = M(n, a) := \sum_{j=0}^a \binom{n-1+j}{j}$$

and from this it follows that for any  $F \in \widetilde{W}_q^{a,p}(\mathbb{R}^N)$  and  $G \in \widetilde{W}_q^{-a,p'}(\mathbb{R}^N)$  we have

$$(42) \quad |\langle F, G \rangle_{\widetilde{L}}| \leq \|F\|_{\widetilde{W}_q^{a,p}} \|G\|_{\widetilde{W}_q^{-a,p'}}$$

see again [A]. Note that since

$$\tilde{C}_q^{\infty,c}(\mathbb{R}^N) \cap C_q^{\infty,c}(\mathbb{R}^N) = \{0\}$$

the spaces  $\tilde{L}_q^r(\mathbb{R}^N)$  and  $L_q^r(\mathbb{R}^N)$  are transversal; the same is true for  $\tilde{W}_q^{a,p}(\mathbb{R}^N)$  and  $W_q^{a,p}(\mathbb{R}^N)$  and for the respective dual spaces.

**2.3. Operators and their adjoints.** For  $\aleph$  as in (30) and for any admissible degree increment  $\ell$ , we define a  $k$ th-order differential condition via the action

$$(43) \quad \sum_{I \in \mathcal{I}(N,q)} F_I dz^I \rightarrow \sum_{L \in \mathcal{I}(N,q+\ell)} \left( \sum_{\substack{I \in \mathcal{I}(N,q) \\ \alpha \in \mathcal{S}(n,k)}} \epsilon_L^{\aleph(i\alpha)I} \frac{\partial^k F_I}{\partial z^{i\alpha}} \right) dz^L$$

where  $N$  is as in (23) and (24) and  $q \in \{0, 1, \dots, N\}$ . Here  $i\alpha$  is as in (28) and  $\aleph$  is the correspondence (30). This action produces a differential operator  $T_{\ell,\aleph}$  that maps

$$(44) \quad T_{\ell,\aleph} : C_q^{\infty,c}(\mathbb{R}^N) \rightarrow C_{q+\ell}^{\infty,c}(\mathbb{R}^N), \quad 0 \leq q \leq N.$$

It follows from (35) that the action (43) also determines an operator

$$(45) \quad \tilde{T}_{\ell,\aleph} : \tilde{C}_q^{\infty,c}(\mathbb{R}^N) \rightarrow \tilde{C}_{q+\ell}^{\infty,c}(\mathbb{R}^N), \quad 0 \leq q \leq N.$$

Now observe that (20) grants that the pullback by  $\pi$  maps

$$\pi^* : C_q^{\infty,c}(\mathbb{R}^n) \rightarrow \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$$

see (32). On the other other hand, it is immediate to check that

$$i^* : \tilde{C}_q^{\infty,c}(\mathbb{R}^N) \rightarrow C_q^{\infty,c}(\mathbb{R}^n).$$

On account of these observations we see that the action (43) produces a third operator  $\mathcal{T}_{\ell,\aleph}$  that maps

$$(46) \quad \mathcal{T}_{\ell,\aleph} : C_q^{\infty,c}(\mathbb{R}^n) \rightarrow C_{q+\ell}^{\infty,c}(\mathbb{R}^n), \quad 0 \leq q \leq n$$

and is defined as follows

$$(47) \quad \mathcal{T}_{\ell,\aleph} := i^* \tilde{T}_{\ell,\aleph} \pi^*.$$

Note that  $\mathcal{T}_{\ell,\aleph}$  acts non-trivially only for

$$(48) \quad n \geq \ell.$$

Condition (48) may be viewed in two different ways: as a constraint on the size of the degree increment  $\ell$  relative to the pair  $(n, k)$  (however note that (48) is satisfied by  $\ell = 1$  for *any* pair  $(n, k)$ ) or as a constraint on the size of  $n$  relative to  $k$  (and in this case, imposing the constraint  $n \geq k$  ensures that (48) holds for *all* admissible degree increments).

In the following,  $\langle \cdot, \cdot \rangle$  denotes the duality in  $W_q^{p,2}(\mathbb{R}^N)$  (resp.  $W_q^{p,2}(\mathbb{R}^n)$ ).

**Proposition 2.2.** *Let  $\ell$  be an admissible degree increment for  $(n, k)$ . The formal adjoint of  $T_{\ell, \mathbb{N}}$  on  $W_q^{a,2}(\mathbb{R}^N)$  is*

$$(49) \quad T_{\ell, \mathbb{N}}^* := (-1)^{k+q(N-\ell-q)} *_N T_{\ell, \mathbb{N}} *_N, \quad 0 \leq q \leq N$$

*That is, for any  $F \in C_q^{\infty, c}(\mathbb{R}^N)$  and for any  $G \in C_{q+\ell}^{\infty, c}(\mathbb{R}^N)$  we have*

$$(50) \quad \langle T_{\ell, \mathbb{N}} F, G \rangle = \langle F, T_{\ell, \mathbb{N}}^* G \rangle$$

*The formal adjoint of  $\widetilde{T}_{\ell, \mathbb{N}}$  on  $\widetilde{W}_q^{a,2}(\mathbb{R}^N)$  is*

$$(51) \quad \widetilde{T}_{\ell, \mathbb{N}}^* := (-1)^{k+q(N-\ell-q)} *_N \widetilde{T}_{\ell, \mathbb{N}} *_N, \quad 0 \leq q \leq N$$

*That is, for any  $F \in \widetilde{C}_q^{\infty, c}(\mathbb{R}^N)$  and for any  $G \in \widetilde{C}_{q+\ell}^{\infty, c}(\mathbb{R}^N)$  we have*

$$(52) \quad \langle \widetilde{T}_{\ell, \mathbb{N}} F, G \rangle_{\widetilde{L}} = \langle F, \widetilde{T}_{\ell, \mathbb{N}}^* G \rangle_{\widetilde{L}}$$

*Suppose further that  $n \geq \ell$ . Then, the formal adjoint of  $\mathcal{T}_{\ell, \mathbb{N}}$  on  $W_q^{a,2}(\mathbb{R}^n)$  is*

$$(53) \quad \mathcal{T}_{\ell, \mathbb{N}}^* := (-1)^{k+q(n-\ell-q)} *_n \mathcal{T}_{\ell, \mathbb{N}} *_n, \quad 0 \leq q \leq n$$

*that is, for any  $f \in C_q^{\infty, c}(\mathbb{R}^n)$  and for any  $g \in C_{q+\ell}^{\infty, c}(\mathbb{R}^n)$  we have*

$$(54) \quad \langle \mathcal{T}_{\ell, \mathbb{N}} f, g \rangle = \langle f, \mathcal{T}_{\ell, \mathbb{N}}^* g \rangle.$$

## 2.4. Estimates.

**Theorem 2.3.** *Let  $n \geq 2$  and  $k \geq 1$  be given. Let  $\ell \in \{1, \dots, k\}$  be any admissible degree increment for the pair  $(n, k)$ , and let  $N$  be a solution of (23) that satisfies (24).*

*For any  $0 \leq q \leq N - \ell$  and for any  $F \in \widetilde{L}_q^1(\mathbb{R}^N)$ , if*

$$(55) \quad \widetilde{T}_{\ell, \mathbb{N}} F = 0$$

*in the sense of distributions, then*

$$(56) \quad |\langle F, H \rangle_{\widetilde{L}}| \leq \widetilde{C} \|F\|_{\widetilde{L}_q^1} \|\nabla H\|_{\widetilde{L}_q^n}$$

*for any  $H \in (\widetilde{L}_q^\infty \cap \widetilde{W}_q^{1,n})(\mathbb{R}^N)$ .*

*For any  $\ell \leq p \leq N$  and for any  $G \in \widetilde{L}_p^1(\mathbb{R}^N)$ , if*

$$(57) \quad \widetilde{T}_{\ell, \mathbb{N}}^* G = 0$$

*in the sense of distributions, then*

$$(58) \quad |\langle G, K \rangle_{\widetilde{L}}| \leq \widetilde{C} \|G\|_{\widetilde{L}_p^1} \|\nabla K\|_{\widetilde{L}_p^n}$$

*for any  $K \in (\widetilde{L}_p^\infty \cap \widetilde{W}_p^{1,n})(\mathbb{R}^N)$ .*

The constant  $\tilde{C}$  depends only on  $n$  and  $k$ .

**Theorem 2.4.** *Let  $n \geq 2$  and  $k \geq 1$  be given. Let  $\ell \in \{1, \dots, k\}$  be an admissible degree increment for the pair  $(n, k)$  such that*

$$n \geq \ell.$$

*For any  $0 \leq q \leq n - \ell$  and for any  $f \in L_q^1(\mathbb{R}^n)$ , if*

$$(59) \quad \mathcal{T}_{\ell, \mathbb{N}} f = 0$$

*in the sense of distributions, then*

$$(60) \quad |\langle f, h \rangle| \leq \mathcal{C} \|f\|_{L_q^1} \|\nabla h\|_{L_q^n}$$

*for any  $h \in (L_q^\infty \cap W_q^{1,n})(\mathbb{R}^n)$ .*

*For any  $\ell \leq p \leq n$  and for any  $g \in L_p^1(\mathbb{R}^n)$ , if*

$$(61) \quad \mathcal{T}_{\ell, \mathbb{N}}^* g = 0$$

*in the sense of distributions, then*

$$(62) \quad |\langle g, h \rangle| \leq \mathcal{C} \|g\|_{L_p^1} \|\nabla h\|_{L_p^n}$$

*for any  $h \in (L_p^\infty \cap W_p^{1,n})(\mathbb{R}^n)$ .*

The constant  $\mathcal{C}$  depends only on  $n$  and  $k$ .

We have

Theorem 1.4  $\iff$  Theorem 2.3  $\Rightarrow$  Theorem 2.4  $\Rightarrow$  Theorem 1.3.

**2.5. Hodge systems.** Concerning the compatibility conditions for the Hodge system for each of  $\tilde{T}_{\ell, \mathbb{N}}$  and  $\mathcal{T}_{\ell, \mathbb{N}}$ , we have

$$(\tilde{T}_{\ell, \mathbb{N}} \circ \tilde{T}_{\ell, \mathbb{N}}) F = (1 + (-1)^{\ell^2}) \sum_{M \in \mathcal{I}(N, q+2\ell)} \left( \sum_{\substack{I \in \mathcal{I}(N, q) \\ \alpha, \beta \in \mathcal{S}(n, k)}} \epsilon_M^{\mathbb{N}(i\beta)\mathbb{N}(i\alpha)I} \frac{\partial^{2k} F_I}{\partial z^{i\alpha} \partial z^{i\beta}} \right) dz^M$$

so in particular

$$(63) \quad \tilde{T}_{\ell, \mathbb{N}} \circ \tilde{T}_{\ell, \mathbb{N}} = 0 \iff \ell \text{ is odd.}$$

A similar computation shows that the same is true for  $\mathcal{T}_{\ell, \mathbb{N}}$ , so in the sequel we will often pay special attention to the admissible degree increment  $\ell = 1$ .

**Lemma 2.5.** *Let  $\tilde{T}_{\ell, \aleph}^*$  be given by (51) and set*

$$(64) \quad \tilde{\square}_{\ell, \aleph} := \tilde{T}_{\ell, \aleph} \tilde{T}_{\ell, \aleph}^* + \tilde{T}_{\ell, \aleph}^* \tilde{T}_{\ell, \aleph}.$$

*If*

$$H = \sum_{I \in \mathcal{I}(N, q)} H_I dz^I \in \tilde{C}_q^{\infty, c}(\mathbb{R}^N)$$

*then*

$$(65) \quad \tilde{\square}_{\ell, \aleph} H = (-1)^{k+\ell N} \sum_{\substack{M, I \in \mathcal{I}(N, q) \\ \alpha, \beta \in \mathcal{S}(n, k)}} \tilde{C}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} \frac{\partial^{2k} H_I}{\partial z^{i\alpha} \partial z^{i\beta}} dz^M$$

*where*

$$(66) \quad \tilde{C}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} = \sum_{L \in \mathcal{I}(N, q+\ell)} \epsilon_{\aleph(i\alpha)I}^L \cdot \epsilon_{\aleph(i\beta)M}^L + \sum_{K \in \mathcal{I}(N, q-\ell)} \epsilon_{\aleph(i\alpha)K}^M \cdot \epsilon_{\aleph(i\beta)K}^I.$$

*In particular, for  $\ell = 1$  we have*

$$(67) \quad \tilde{\square}_{1, \aleph} H = \sum_{I \in \mathcal{I}(N, q)} (\tilde{\square}_{1, \aleph} H_I) dz^I = (-1)^{k+N} \sum_{\substack{I \in \mathcal{I}(N, q) \\ \alpha \in \mathcal{S}(n, k)}} \frac{\partial^{2k} H_I}{\partial z^{i\alpha} \partial z^{i\alpha}} dz^I.$$

*Let  $\mathcal{T}_{\ell, \aleph}$  be given by (53) (assume  $n \geq \ell$ ) and set*

$$(68) \quad \square_{\ell, \aleph} := \mathcal{T}_{\ell, \aleph} \mathcal{T}_{\ell, \aleph}^* + \mathcal{T}_{\ell, \aleph}^* \mathcal{T}_{\ell, \aleph}.$$

*If*

$$h = \sum_{I \in \mathcal{I}(n, q)} h_I dz^I \in C_q^{\infty, c}(\mathbb{R}^n)$$

*then*

$$(69) \quad \square_{\ell, \aleph} h = (-1)^{k+\ell n} \sum_{\substack{M, I \in \mathcal{I}(n, q) \\ \alpha, \beta \in (\pi \circ \aleph^{-1})(\mathcal{I}(n, \ell))}} C_{\aleph(i\alpha) \aleph(i\beta)}^{MI} \frac{\partial^{2k} h_I}{\partial x^\alpha \partial x^\beta} dx^M$$

*where*

$$(70) \quad C_{\aleph(i\alpha) \aleph(i\beta)}^{MI} = \sum_{L \in \mathcal{I}(n, q+\ell)} \epsilon_{\aleph(i\alpha)I}^L \cdot \epsilon_{\aleph(i\beta)M}^L + \sum_{K \in \mathcal{I}(n, q-\ell)} \epsilon_{\aleph(i\alpha)K}^M \cdot \epsilon_{\aleph(i\beta)K}^I.$$

*In particular, for  $\ell = 1$  we have*

$$(71) \quad \square_{1, \aleph} h = \sum_{I \in \mathcal{I}(n, q)} (\square_{1, \aleph} h_I) dx^I = (-1)^{k+n} \sum_{\substack{I \in \mathcal{I}(n, q) \\ \alpha \in (\pi \circ \aleph^{-1})(\{1, \dots, n\})}} \frac{\partial^{2k} h_I}{\partial x^\alpha \partial x^\alpha} dx^I.$$

**Corollary 2.6.** *For any  $0 \leq q \leq N$  and for any choice of the correspondence  $\aleph$ , the operator  $\tilde{\square}_{1,\aleph}$  satisfies the Legendre-Hadamard condition in the following sense:*

$$(72) \quad \operatorname{Re} \left( \sum_{\substack{I, M \in \mathcal{I}(N, q) \\ \alpha, \beta \in \mathcal{S}(n, k)}} \tilde{C}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} \xi^{i\alpha} \xi^{i\beta} \zeta_I \bar{\zeta}_M \right) \geq C |\xi|^{2k} |\zeta|^2$$

for any  $\xi \in i(\mathbb{R}^n)$  and for any  $\zeta \in \mathbb{C}^{\binom{N}{q}}$ .

See [KPV]. Indeed, by (67) we have that the coordinate-based representation of  $\tilde{\square}_{1,\aleph}$  is independent of the choice of  $\aleph$  and furthermore

$$\operatorname{Re} \left( \sum_{\substack{I, M \in \mathcal{I}(N, q) \\ \alpha, \beta \in \mathcal{S}(n, k)}} \tilde{C}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} \xi^{i\alpha} \xi^{i\beta} \zeta_I \bar{\zeta}_M \right) = \left( \sum_{\alpha \in \mathcal{S}(n, k)} \xi_1^{2\alpha_1} \dots \xi_n^{2\alpha_n} \right) |\zeta|^2$$

and if  $\xi \in i(\mathbb{R}^n)$  then

$$\sum_{\alpha \in \mathcal{S}(n, k)} \xi_1^{2\alpha_1} \dots \xi_n^{2\alpha_n} \geq C |\xi|^{2k}$$

where  $C = C(n, k)$ . On the other hand, the coordinate-based representation of  $\square_{1,\aleph}$  does depend on the choice of  $\aleph$ , see (67), and so does the uniform ellipticity of  $\square_{1,\aleph}$ ; for instance, if  $\aleph$  is chosen so that

$$(\pi \circ \aleph^{-1})(\{1, 2, \dots, n\}) =$$

$$= \{(k, 0, \dots, 0), (1, k-1, 0, \dots, 0), \dots, (1, 0, \dots, 0, k-1)\} \subset \mathcal{S}(n, k)$$

then  $\square_{1,\aleph}$  has symbol

$$\left( \sum_{j=1}^n \xi_1^2 \xi_j^{2(k-1)} \right) |\zeta|^2, \quad \xi \in \mathbb{R}^n, \quad \eta \in \mathbb{C}^{\binom{n}{q}}$$

which fails to be uniformly elliptic (take e.g.,  $\xi := (0, 1, \dots, 1)$ ). Choosing instead

$$(\pi \circ \aleph^{-1})(\{1, 2, \dots, n\}) =$$

$$\{(k, 0, \dots, 0), (0, k, 0, \dots, 0), \dots, (0, \dots, 0, k)\}$$

(corresponding to the example  $\mathcal{T}$  discussed in the Introduction) leads to an operator  $\square_{1,\aleph}$  which is easily verified to be uniformly elliptic, as we have

$$\sum_{j=1}^n \xi_j^{2k} \geq n^{1-k} |\xi|^{2k}.$$

**Lemma 2.7.** *We have that*

$$\tilde{\square}_{1,\aleph}: \tilde{C}_q^{\infty,c}(\mathbb{R}^N) \rightarrow \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$$

*is invertible for any  $1 < p < \infty$ . For any  $\varphi \in \tilde{L}_q^p(\mathbb{R}^N)$  we have*

$$(73) \quad \|\Phi\|_{\tilde{W}_q^{2k,p}} \lesssim \|\varphi\|_{\tilde{L}_q^p}$$

*where  $\Phi := \tilde{\square}_{1,\aleph}^{-1} \varphi$ .*

**Theorem 2.8.** *Suppose that  $F \in \tilde{L}_{q+1}^1(\mathbb{R}^N)$  and  $G \in \tilde{L}_{q-1}^1(\mathbb{R}^N)$  satisfy the hypotheses of Theorem 2.3. Let*

$$(74) \quad Z = \tilde{\square}_{1,\aleph}^{-1}(\tilde{T}_{1,\aleph}^* F + \tilde{T}_{1,\aleph} G) \in \tilde{\Lambda}_q(\mathbb{R}^N), \quad 0 \leq q \leq N$$

*be the solution of the Hodge system for  $\tilde{T}_{1,\aleph}$  with data  $(F, G)$ , that is:*

$$(75) \quad \begin{cases} \tilde{T}_{1,\aleph} Z = F \\ \tilde{T}_{1,\aleph}^* Z = G \end{cases}$$

*Then*

$$(76) \quad \|Z\|_{\tilde{W}_q^{k-1,r}} \leq C(\|F\|_{\tilde{L}_{q+1}^1} + \|G\|_{\tilde{L}_{q-1}^1}), \quad \text{for } r = n/(n-1)$$

*whenever  $q$  is neither 1 (unless  $G = 0$ ) nor  $N-1$  (unless  $F = 0$ ).*

We have:

$$\text{Theorem 2.3 } (\ell = 1; \quad 1 \leq p, q \leq N-1) \iff \text{Theorem 2.8}$$

For those choices of  $\aleph$  that give rise to a uniformly elliptic  $\square_{1,\aleph}$ , an analogous result holds for

$$(77) \quad \begin{cases} \mathcal{T}_{1,\aleph} h = f, & \mathcal{T}_{1,\aleph} f = 0 \\ \mathcal{T}_{1,\aleph}^* h = g, & \mathcal{T}_{1,\aleph}^* g = 0 \end{cases}$$

which turns out to be equivalent to Theorem 2.4 ( $\ell = 1$ ). We omit the statement.

We remark in closing that for  $\ell \geq 2$  there is no analog of (67). Indeed, setting

$$\{\lambda_0\} := \{\aleph(i\alpha)\} \cap \{\aleph(i\beta)\}$$

and

$$\widehat{\{\aleph(i\alpha)\}} := \{\aleph(i\alpha)\} \setminus \{\lambda_0\}; \quad \widehat{\{\aleph(i\beta)\}} := \{\aleph(i\beta)\} \setminus \{\lambda_0\}$$



(where the brackets  $\{ \}$  indicate that the (ordered) label  $J$  is being regarded as an (unordered) set  $\{J\}$ ), it can be proved that

$$(78) \quad \tilde{C}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = (1 + (-1)^{(\ell - |\{\lambda_0\}|)^2}) \epsilon_{\lambda_0 \aleph(i\alpha)}^{\aleph(i\alpha)} \cdot \epsilon_{\lambda_0 \aleph(i\beta)}^{\aleph(i\beta)} \cdot \epsilon_{\aleph(i\beta)M}^{\aleph(i\alpha)I}.$$

In particular, the coordinate-based representation of  $\tilde{\square}_{\ell, \aleph}$  does depend on the choice of the representation  $\aleph$ , and it is no longer true that  $\tilde{C}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = 0$  whenever  $\alpha \neq \beta$ , even for odd  $\ell$ .

### 3. PROOFS

*Proof of Lemma 2.1.* Conclusions *i.* and *ii.* are an immediate consequence of the (classical) theory for  $\mathbb{R}^n$  combined with the readily verified identities:

$$(79) \quad \|F\|_{\tilde{L}_q^r}^r = \sum_{I \in \mathcal{I}(N, q)} \int_{\mathbb{R}^n} \left( \sum_{I \in \mathcal{I}(N, q)} |F_I \circ i|^2(x) \right)^{r/2} dV(x)$$

and

$$(80) \quad \langle F, G \rangle_{\tilde{L}} = \sum_{I \in \mathcal{I}(N, q)} \int_{\mathbb{R}^n} (F_I \circ i)(x) \cdot (G_I \circ i)(x) dV(x).$$

To prove the density of  $\tilde{C}_q^{\infty, c}(\mathbb{R}^N)$  in  $\tilde{L}_q^r(\mathbb{R}^N)$ , let

$$F = \sum_{I \in \mathcal{I}(N, q)} F_I dz^I \in \tilde{L}_q^r(\mathbb{R}^N)$$

be given. By the definition of  $\tilde{L}_q^r(\mathbb{R}^N)$ , for any  $I \in \mathcal{I}(N, q)$  we have that  $F_I \circ i \in L^r(\mathbb{R}^n)$ , and so there is  $\{f_{j,I}\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$  such that

$$(81) \quad \|f_{j,I} - F_I \circ i\|_{L^r(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } j \rightarrow \infty.$$

Define

$$F_j = \sum_{I \in \mathcal{I}(N, q)} F_{j,I} dz^I, \quad F_{j,I} := f_{j,I} \circ \pi$$

Then, using (20), we see that

$$F_{j,I} \circ i \circ \pi = F_{j,I} \quad \text{and} \quad F_{j,I} \circ i = f_{j,I} \in C_c^\infty(\mathbb{R}^n)$$

hold for any  $I \in \mathcal{I}(N, q)$ , and from these it follows that

$$\{F_j\}_{j \in \mathbb{N}} \subset \tilde{C}_q^{\infty, c}(\mathbb{R}^N).$$

Moreover, on account of (79) and (81), there is  $C = C(r, N)$  such that

$$\|F_j - F\|_{\tilde{L}_q^r}^r \leq C \sum_{I \in \mathcal{I}(N, q)} \|f_{j,I} - F_I \circ i\|_{L^r(\mathbb{R}^n)}^r \rightarrow 0 \quad \text{as } j \rightarrow \infty,$$

as desired. The conclusions concerning the Sobolev spaces follow from the theory for  $W_q^{a,p}(\mathbb{R}^n)$  via (34).  $\square$

*Proof of Proposition 2.2.* Let  $F \in C_q^{\infty,c}(\mathbb{R}^N)$  and  $G \in C_{q+\ell}^{\infty,c}$  be given. One has

$$T_{\ell,\mathbb{N}}F \wedge *_N G = \sum_{L \in \mathcal{I}(N, q+\ell)} \left( \sum_{\substack{I \in \mathcal{I}(N, q) \\ \alpha \in \mathcal{S}(n, k)}} \epsilon_L^{\mathbb{N}(i\alpha)I} \frac{\partial^k F_I}{\partial z^{i\alpha}} G_L \right) dV$$

Integrating both sides of this identity and then further integrating the right-hand side by parts  $k$ -many times we find

$$(82) \quad \langle T_{\ell,\mathbb{N}}F, G \rangle = (-1)^k \int_{\mathbb{R}^N} \sum_{L \in \mathcal{I}(N, q+\ell)} \left( \sum_{\substack{I \in \mathcal{I}(N, q) \\ \alpha \in \mathcal{S}(n, k)}} \epsilon_L^{\mathbb{N}(i\alpha)I} F_I \frac{\partial^k G_L}{\partial z^{i\alpha}} \right) dV$$

On the other hand, a computation that requires manipulating the coefficients  $\epsilon_J^{\mathbb{N}(i\alpha)K}$  gives

$$F \wedge *_N (*_N T_{\ell,\mathbb{N}} *_N G) = (-1)^{q(N-q)+q\ell} \sum_{L \in \mathcal{I}(N, q+\ell)} \left( \sum_{\substack{I \in \mathcal{I}(N, q) \\ \alpha \in \mathcal{S}(n, k)}} \epsilon_L^{\mathbb{N}(i\alpha)I} F_I \frac{\partial^k G_L}{\partial z^{i\alpha}} \right) dV$$

Identity (50) is now obtained by integrating the two sides of the identity above and comparing with (82) after having adjusted the multiplicative constants as in (49). Note that since

$$D^\lambda T_{\ell,\mathbb{N}}F = T_{\ell,\mathbb{N}} D^\lambda F \quad \text{for any multi-index } \lambda$$

where  $D^\lambda F \in C_q^{\infty,c}(\mathbb{R}^N)$  is defined as in (39), the same argument also shows that

$$\langle D^\lambda T_{\ell,\mathbb{N}}F, D^\lambda G \rangle = \langle D^\lambda F, D^\lambda T_{\ell,\mathbb{N}}^* G \rangle$$

The proofs of (52) and of (54) follow in a similar fashion.  $\square$

*Theorem 1.4  $\Rightarrow$  Theorem 2.3.* Let  $\ell$  be an admissible degree increment and let  $0 \leq q \leq N - \ell$ . Suppose that

$$F = \sum_{I \in \mathcal{I}(N, q)} F_I dz^I \in \tilde{\Lambda}_q(\mathbb{R}^N)$$

and

$$H = \sum_{I \in \mathcal{I}(N, q)} H_I dz^I \in \tilde{\Lambda}_q(\mathbb{R}^N)$$

satisfy the hypotheses of Theorem 2.3. Fix an arbitrary  $I_0 \in \mathcal{I}(N, q)$ , and choose (any)  $L_0 \in \mathcal{I}(N, q + \ell)$  so that

$$I_0 \subseteq L_0.$$

(The hypothesis:  $q \leq N - \ell$  grants  $q + \ell \leq N$  and so at least one such  $L_0$  must exist.) With  $I_0$  and  $L_0$  fixed as above, define  $h^{L_0} = (h_\alpha^{L_0})_{\alpha \in \mathcal{S}(n,k)}$  and  $g^{L_0} = (g_\alpha^{L_0})_{\alpha \in \mathcal{S}(n,k)}$  via

$$\begin{aligned} h_\alpha^{L_0} &= \sum_{I \in \mathcal{I}(N,q)} \epsilon_{L_0}^{\aleph(i\alpha)I} H_I \circ i, & \alpha \in \mathcal{S}(n,k); \\ g_\alpha^{L_0} &= \sum_{I \in \mathcal{I}(N,q)} \epsilon_{L_0}^{\aleph(i\alpha)I} F_I \circ i, & \alpha \in \mathcal{S}(n,k). \end{aligned}$$

We claim that  $g^{L_0}$  satisfies condition (12) in Theorem 1.4: to this end, note that by (35) we have

$$\sum_{\alpha \in \mathcal{S}(n,k)} \frac{\partial^k g_\alpha^{L_0}}{\partial x^\alpha} = \left( \sum_{\substack{I \in \mathcal{I}(N,q) \\ \alpha \in \mathcal{S}(n,k)}} \epsilon_{L_0}^{\aleph(i\alpha)I} \frac{\partial^k F_I}{\partial z^{i\alpha}} \right) \circ i = [T_\ell, \aleph F]_{L_0} \circ i = 0$$

where the last identity is due to the hypothesis (55). It thus follow from Theorem 1.4 that

$$\left| \int_{\mathbb{R}^n} g_{\alpha_0}^{L_0} \cdot h_{\alpha_0}^{L_0} \right| \leq C \|g^{L_0}\|_{L^1(\mathbb{R}^n)} \|\nabla h^{L_0}\|_{L^n(\mathbb{R}^n)}$$

where  $\alpha_0 \in \mathcal{S}(n,k)$  is uniquely determined by  $I_0$  and  $L_0$  via

$$i\alpha_0 := \aleph^{-1}(L_0 \setminus I_0),$$

(note that  $L_0 \setminus I_0 \in \mathcal{I}(N, \ell)$ .) But for  $\alpha_0$  as above we have

$$\epsilon_{L_0}^{\aleph(i\alpha_0)I} \neq 0 \iff I = I_0$$

and so

$$g_{\alpha_0}^{L_0} = \epsilon_{L_0}^{\aleph(i\alpha_0)I_0} F_{I_0} \circ i, \quad \text{and} \quad h_{\alpha_0}^{L_0} = \epsilon_{L_0}^{\aleph(i\alpha_0)I_0} H_{I_0} \circ i.$$

On the other hand, it is immediate to verify that

$$\|g^{L_0}\|_{L^1(\mathbb{R}^n)} \lesssim \|F\|_{\tilde{L}_q^1(\mathbb{R}^N)}, \quad \|\nabla h^{L_0}\|_{L^n(\mathbb{R}^n)} \lesssim \|\nabla H\|_{\tilde{L}_q^n(\mathbb{R}^N)},$$

and

$$\int_{\mathbb{R}^n} g_{\alpha_0}^{L_0} \cdot h_{\alpha_0}^{L_0} = \int_{\mathbb{R}^n} (F_{I_0} \circ i) \cdot (H_{I_0} \circ i)(x) dV(x).$$

Since  $I_0 \in \mathcal{I}(N, q)$  had been fixed arbitrarily, we have proved that

$$(83) \quad \left| \int_{\mathbb{R}^n} \left( (F_I \circ i) \cdot (H_I \circ i) \right)(x) dV(x) \right| \leq C \|F\|_{\tilde{L}_q^1(\mathbb{R}^N)} \|\nabla H\|_{\tilde{L}_q^n(\mathbb{R}^N)}$$

holds for any  $I \in \mathcal{I}(N, q)$ , for any  $0 \leq q \leq N - \ell$  and for any admissible degree increment  $\ell$ . Inequality (56) follows from (83) and the

coordinate-based representation for  $\langle \cdot, \cdot \rangle_{\tilde{L}}$ , see (80). (We remark that in the special case  $q = 0$ , the proof follows along these very same lines by defining  $g_\alpha := F \circ i$  for each  $\alpha \in \mathcal{S}(n, k)$ .)

In order to prove (58), it suffices to apply (56) to:  $F := *_N G$  and  $H := *_N K$  (with  $q := N - p$ ).  $\square$

*Theorem 2.3  $\Rightarrow$  Theorem 1.4.* Let  $\ell$  be any admissible degree increment for  $(n, k)$  and let  $\aleph$  be any one-to-one correspondence:  $i\mathcal{S}(n, k) \rightarrow \mathcal{I}(N, \ell)$ . Suppose that  $g$  and  $h$  satisfy the hypotheses of Theorem 1.4; without loss of generality we may assume that  $g_\alpha, h_\alpha \in C_0^\infty(\mathbb{R}^n)$ ,  $\alpha \in \mathcal{S}(n, k)$ . Choose  $q := N - \ell$  and define  $F$  and  $H$  in  $\Lambda_{N-\ell}(\mathbb{R}^N)$  via

$$(84) \quad F_I := \epsilon_{(1, \dots, N)}^{I'I} g_\alpha \circ \pi, \quad I \in \mathcal{I}(N, N - \ell)$$

$$(85) \quad H_I := \epsilon_{(1, \dots, N)}^{I'I} h_\alpha \circ \pi, \quad I \in \mathcal{I}(N, N - \ell)$$

where  $I' := \{1, \dots, N\} \setminus I \in \mathcal{I}(N, \ell)$ , and  $\alpha \in \mathcal{S}(n, k)$  is uniquely determined by  $I$  and by  $\aleph$  via

$$i\alpha = \aleph^{-1}(I').$$

Since  $\pi \circ i$  is the identity on  $\mathbb{R}^n$ , we have

$$F_I \circ i \circ \pi = F_I, \quad F_I \circ i = \epsilon_{(1, \dots, N)}^{I'I} g_\alpha \in C_0^\infty(\mathbb{R}^n)$$

so  $F \in \tilde{C}_{N-\ell}^{\infty, c}(\mathbb{R}^N)$  and

$$\tilde{T}_{\ell, \aleph} F = [\tilde{T}_{\ell, \aleph} F]_N dz_1 \wedge \dots \wedge dz_N \in \tilde{C}_N^{\infty, c}(\mathbb{R}^N).$$

Using (35) and (84) we find

$$\begin{aligned} [\tilde{T}_{\ell, \aleph} F]_N \circ i &= \left( \sum_{\alpha \in \mathcal{S}(n, k)} \epsilon_{1, \dots, N}^{\aleph(i\alpha) \aleph(i\alpha)'} \frac{\partial^k}{\partial z^{i\alpha}} F_{\aleph(i\alpha)'} \right) \circ i = \\ &= \sum_{\alpha \in \mathcal{S}(n, k)} \frac{\partial^k g_\alpha}{\partial x^\alpha} = 0. \end{aligned}$$

where the last identity is due to the hypothesis (12). Now observe that if  $G \in \tilde{\Lambda}_q(\mathbb{R}^N)$  then

$$G = 0 \iff G_I \circ i = 0 \quad \text{for each } I \in \mathcal{I}(N, q).$$

Combining all of the above we obtain

$$\tilde{T}_{\ell, \aleph} F = 0$$

so that Theorem 2.3 grants

$$|\langle F, H \rangle_{\tilde{L}}| \leq \tilde{C} \|F\|_{\tilde{L}_q^1(\mathbb{R}^N)} \|\nabla H\|_{\tilde{L}_q^n(\mathbb{R}^N)}.$$

But since  $(\pi \circ i)(x) = x$  for all  $x \in \mathbb{R}^n$  it follows from (80), (84) and (85) that

$$\langle F, H \rangle_{\tilde{L}} = \sum_{\alpha \in \mathcal{S}(n,k)} \int g_\alpha \cdot h_\alpha; \quad \|F\|_{\tilde{L}_q^1(\mathbb{R}^N)} = \|g\|_{L^1}; \quad \|\nabla H\|_{\tilde{L}_q^n(\mathbb{R}^N)} = \|\nabla h\|_n$$

and so

$$(86) \quad \left| \sum_{\alpha \in \mathcal{S}(n,k)} \int g_\alpha \cdot h_\alpha \right| \leq \tilde{C} \|g\|_{L^1} \|\nabla h\|_{L^n}$$

is true for any  $h \in (L^\infty \cap W^{1,n})(\mathbb{R}^n, \mathbb{R}^m)$ . Now fix  $\alpha_0 \in \mathcal{S}(n, k)$  arbitrarily and define

$$\hat{h}_\alpha := \delta_{\alpha_0 \alpha} h_\alpha, \quad \alpha \in \mathcal{S}(n, k)$$

where  $\delta_{\alpha_0 \alpha}$  denotes the Kroenecker symbol. Then  $\hat{h} \in (L^\infty \cap W^{1,n})(\mathbb{R}^n, \mathbb{R}^m)$  and so by applying (86) to  $\hat{h}$  we obtain

$$\left| \sum_{\alpha \in \mathcal{S}(n,k)} \int g_\alpha \cdot \hat{h}_\alpha \right| \leq \tilde{C} \|g\|_{L^1} \|\nabla \hat{h}\|_{L^n}.$$

However

$$\left| \sum_{\alpha \in \mathcal{S}(n,k)} \int g_\alpha \cdot \hat{h}_\alpha \right| = \left| \int g_{\alpha_0} \cdot h_{\alpha_0} \right| \quad \text{and} \quad \|\nabla \hat{h}\|_{L^n} \leq \|\nabla h\|_{L^n},$$

so (13) is true for any choice of  $\alpha_0 \in \mathcal{S}(n, k)$ , with  $C := \tilde{C}$ .  $\square$

*Theorem 2.3  $\Rightarrow$  Theorem 2.4.* Let  $\ell$  be an admissible degree increment such that

$$n \geq \ell,$$

let  $\aleph$  be any one-to-one correspondence:  $i\mathcal{S}(n, k) \rightarrow \mathcal{I}(N, \ell)$  and let  $0 \leq q \leq n - \ell$  be given. Suppose that

$$f = \sum_{I \in \mathcal{I}(n,q)} f_I dx^I \in L_q^1(\mathbb{R}^n)$$

satisfies the hypotheses of Theorem 2.4; without loss of generality we may assume that  $f \in C_q^{\infty,c}(\mathbb{R}^n)$ . By the definition of  $\mathcal{T}_{\ell,\aleph}$ , see (47), we

have

$$\mathcal{T}_{\ell, \aleph} f = \sum_{L \in \mathcal{I}(n, q + \ell)} \left( \sum_{\substack{I \in \mathcal{I}(n, q) \\ \aleph(i\alpha) \in \mathcal{I}(n, \ell)}} \epsilon_L^{\aleph(i\alpha)I} \left( \frac{\partial^k (f_I \circ \pi)}{\partial z^{i\alpha}} \right) \circ i \right) dx^L$$

and applying (35) we obtain

$$\mathcal{T}_{\ell, \aleph} f = \sum_{L \in \mathcal{I}(n, q + \ell)} \left( \sum_{\substack{I \in \mathcal{I}(n, q) \\ \aleph(i\alpha) \in \mathcal{I}(n, \ell)}} \epsilon_L^{\aleph(i\alpha)I} \frac{\partial^k f_I}{\partial x^\alpha} \right) dx^L = 0$$

where the last identity is due to the hypothesis (59). Fix  $I_0 \in \mathcal{I}(n, q)$  and choose (any)  $L_0 \in \mathcal{I}(n, q + \ell)$  so that

$$I_0 \subseteq L_0.$$

(The hypothesis  $q \leq n - \ell$  grants  $q + \ell \leq n$ , so at least one such  $L_0$  must exist.)

Note that since  $\ell \leq n \leq N$  we have  $\mathcal{I}(n, \ell) \subseteq \mathcal{I}(N, \ell)$ , so with  $I_0$  and  $L_0$  fixed as above, we may define

$$F^{L_0} = \sum_{J \in \mathcal{I}(N, \ell)} F_J^{L_0} dz^J \in \tilde{C}_\ell^{\infty, c}(\mathbb{R}^N)$$

via

$$F_J^{L_0} := \begin{cases} \sum_{I \in \mathcal{I}(n, q)} \epsilon_{L_0}^{JI} f_I \circ \pi & \text{for } J \in \mathcal{I}(n, \ell) \\ 0 & \text{for } J \in \mathcal{I}(N, \ell) \setminus \mathcal{I}(n, \ell). \end{cases}$$

Applying (51) with  $q := \ell$  we obtain (ignore the factor  $(-1)^{k+q(N-\ell-q)}$ )

$$\tilde{T}_{\ell, \aleph}^* F^{L_0} = \sum_{\alpha \in \mathcal{S}(n, k)} \frac{\partial^k F_{\aleph(i\alpha)}^{L_0}}{\partial z^{i\alpha}} \in \tilde{C}_0^{\infty, c}(\mathbb{R}^N)$$

and by the definition of  $F^{L_0}$  this is further simplified to

$$\tilde{T}_{\ell, \aleph}^* F^{L_0} = \sum_{\substack{I \in \mathcal{I}(n, q) \\ \aleph(i\alpha) \in \mathcal{I}(n, \ell)}} \epsilon_{L_0}^{\aleph(i\alpha)I} \frac{\partial^k (f_I \circ \pi)}{\partial z^{i\alpha}}.$$

Note that on account of (20) and of (35) we have

$$\left( \tilde{T}_{\ell, \aleph}^* F^{L_0} \right) \circ i = \sum_{\substack{I \in \mathcal{I}(n, q) \\ \aleph(i\alpha) \in \mathcal{I}(n, \ell)}} \epsilon_{L_0}^{\aleph(i\alpha)I} \frac{\partial^k f_I}{\partial x^\alpha} = [\mathcal{T}_{\ell, \aleph} f]_{L_0} = 0.$$

But  $\tilde{T}_{\ell, \aleph}^* F^{L_0} \in \tilde{\Lambda}_0(\mathbb{R}^N)$  and so

$$\left( \tilde{T}_{\ell, \aleph}^* F^{L_0} \right) \circ i = 0 \iff \tilde{T}_{\ell, \aleph}^* F^{L_0} = 0,$$

see (19). Thus

$$\tilde{T}_{\ell, \mathbb{N}}^* F^{L_0} = 0$$

and by Theorem 2.3 we conclude that

$$(87) \quad |\langle F^{L_0}, H \rangle_{\tilde{L}}| \leq \tilde{C} \|F^{L_0}\|_{\tilde{L}_\ell^1} \|\nabla H\|_{\tilde{L}_\ell^n}$$

is true for any  $H \in \tilde{C}_\ell^{\infty, c}(\mathbb{R}^N)$ . Now set

$$J_0 := L_0 \setminus I_0 \in \mathcal{I}(n, \ell)$$

and let

$$h = \sum_{I \in \mathcal{I}(n, q)} h_I dx^I \in (L_q^\infty \cap W_q^{1, n})(\mathbb{R}^n)$$

be given (without loss of generality we may assume that  $h \in C_q^{\infty, c}(\mathbb{R}^n)$ ). Define

$$\hat{H} = \sum_{J \in \mathcal{I}(N, \ell)} \hat{H}_J dz^J \in \Lambda_\ell(\mathbb{R}^N)$$

with

$$\hat{H}_J = \delta_{J_0 J} \sum_{I \in \mathcal{I}(n, q)} \epsilon_{L_0}^{JI} h_I \circ \pi, \quad J \in \mathcal{I}(N, \ell),$$

where  $\delta_{J_0 J}$  is the Kroenecker symbol. Then

$$\hat{H}_J \circ i \circ \pi := \hat{H}_J \quad \text{and} \quad \hat{H}_J \circ i \in C_c^\infty(\mathbb{R}^n), \quad J \in \mathcal{I}(N, \ell)$$

so that

$$\hat{H} \in \tilde{C}_\ell^{\infty, c}(\mathbb{R}^N).$$

Note that

$$\langle F^{L_0}, \hat{H} \rangle_{\tilde{L}} = \int_{\mathbb{R}^n} f_{I_0} \cdot h_{I_0}, \quad \text{and} \quad \|\nabla \hat{H}\|_{\tilde{L}^n(\mathbb{R}^N)} \lesssim \|\nabla h\|_{L^n(\mathbb{R}^n)}.$$

Moreover, by the definition of  $F^{L_0}$  we have

$$\|F^{L_0}\|_{\tilde{L}_\ell^1(\mathbb{R}^N)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}.$$

Thus, applying (87) to  $\hat{H}$  we conclude that

$$(88) \quad \left| \int_{\mathbb{R}^n} f_{I_0} \cdot h_{I_0} \right| \leq \tilde{C} \|f\|_{L^1(\mathbb{R}^n)} \|\nabla h\|_{L^n(\mathbb{R}^n)}$$

is true for any  $I_0 \in \mathcal{I}(n, q)$ , for any  $0 \leq q \leq n - \ell$  and for any  $h \in C_q^{\infty, c}(\mathbb{R}^n)$ , and this in turn implies (60).

In order to prove (62), it suffices to apply (60) to:  $f := *_n g$  (with  $q := n - p$ ).  $\square$

*Theorem 2.4*  $\Rightarrow$  *Theorem 1.3*. We claim that, in fact, Theorem 1.3 is equivalent to the statement for  $\mathcal{T}_{\ell, \aleph}^*$  in Theorem 2.4 in the special case:  $\ell = 1$ ;  $q = 1$  and for specific choices of the ordering  $\aleph$ . Indeed it is easy to see that, for  $\ell = 1$  and  $q = 1$ , (87) and (53) give

$$\mathcal{T}_{1, \aleph}^* f = (-1)^{k+n} \sum_{j=1}^n \frac{\partial^k f_j}{\partial x^{\pi \circ \aleph^{-1}(j)}}, \quad f = \sum_{j=1}^n f_j dx_j \in \Lambda_1(\mathbb{R}^n).$$

Choosing now any ordering  $\aleph : i\mathcal{S}(n, k) \leftrightarrow \mathcal{I}(N, 1)$  such that

$$\pi \circ \aleph^{-1}(j) = (0, \dots, 0, k, 0, \dots, 0), \quad j = 1, \dots, n$$

(where, in the expression above, it is understood that  $k$  occupies the  $j$ -th position) we obtain

$$\mathcal{T}_{1, \aleph}^* f = \sum_{j=1}^n \frac{\partial^k u_j}{\partial x_j^k}, \quad u_j := (-1)^{k+n} f_j, \quad j = 1, \dots, n.$$

The equivalence of the two statements is now apparent.  $\square$

*Proof of Lemma 2.5.* The proof of (65) and (66) is a computation that uses (43) along with the following coordinate-based representation for  $\tilde{T}_{\ell, \aleph}^*$ , which is obtained from (51):

$$\tilde{T}_{\ell, \aleph}^* H = (-1)^{k+N\ell} \sum_{\substack{V \in \mathcal{I}(N, q-\ell) \\ I \in \mathcal{I}(N, q) \\ \beta \in \mathcal{S}(n, k)}} \epsilon_I^{\aleph(i\beta)V} \frac{\partial^k H_I}{\partial z^{i\beta}} dz^V$$

To prove (67) we examine (66) in the case  $\ell = 1$ :

$$(89) \quad \tilde{C}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} = \tilde{A}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} + \tilde{B}_{\aleph(i\alpha) \aleph(i\beta)}^{MI}$$

where

$$(90) \quad \tilde{A}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} := \sum_{L \in \mathcal{I}(N, q+1)} \epsilon_{\aleph(i\alpha)I}^L \cdot \epsilon_{\aleph(i\beta)M}^L,$$

$$(91) \quad \tilde{B}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} := \sum_{K \in \mathcal{I}(N, q-1)} \epsilon_{\aleph(i\alpha)K}^M \cdot \epsilon_{\aleph(i\beta)K}^I$$

and distinguish two cases:  $\alpha \neq \beta$ ; and  $\alpha = \beta$ .

Suppose first that  $\alpha \neq \beta$ . In this case we claim that  $\tilde{C}_{\aleph(i\alpha) \aleph(i\beta)}^{MI} = 0$ . The proof of this claim rests on the following

*Remark 3.1.* The truth value of the following three (combined) conditions on  $\aleph$ ,  $\alpha$ ,  $\beta$ ,  $I$  and  $M$ :

$$(92) \quad \aleph(i\alpha) \notin \{I\}; \quad \aleph(i\beta) \notin \{M\}; \quad \{\aleph(i\alpha)\} \cup \{I\} = \{\aleph(i\beta)\} \cup \{M\}$$



is equivalent<sup>1</sup> to the truth value of

$$(93) \quad \aleph(i\alpha) \in \{M\}; \aleph(i\beta) \in \{I\}; \quad \{M\} \setminus \{\aleph(i\alpha)\} = \{I\} \setminus \{\aleph(i\beta)\}.$$

We postpone the proof of Remark 3.1 and continue with the proof of Lemma 2.5; to this end we claim that if  $\alpha \neq \beta$  then

$$(92) \text{ holds} \iff \tilde{A}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} \neq 0$$

Indeed, since  $\alpha$ ,  $\beta$ ,  $I$  and  $M$  are fixed, the summation that defines  $\tilde{A}_{\aleph(i\alpha)\aleph(i\beta)}^{MI}$ , see(90), consists of at most one term, that is

$$\tilde{A}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = \epsilon_{\aleph(i\beta)M}^{L_0} \cdot \epsilon_{\aleph(i\alpha)I}^{L_0}$$

for at most one choice of  $L_0 \in \mathcal{I}(N, q+1)$ , and it's easy to see that (92) holds if, and only if, there is exactly one choice of  $L_0 \in \mathcal{I}(N, q+1)$  such that  $\epsilon_{\aleph(i\beta)M}^{L_0} \cdot \epsilon_{\aleph(i\alpha)I}^{L_0} \neq 0$  and in such case we have

$$(94) \quad \tilde{A}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = \epsilon_{\aleph(i\alpha)I}^{\aleph(i\beta)M}.$$

Similar considerations grant, again for  $\alpha \neq \beta$ , that we also have

$$(93) \text{ holds} \iff \tilde{B}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} \neq 0$$

and if  $\tilde{B}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} \neq 0$  there is a unique choice of  $K_0 \in \mathcal{I}(N, q-1)$  such that

$$(95) \quad \tilde{B}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = \epsilon_{\aleph(i\alpha)K_0}^M \epsilon_{\aleph(i\beta)K_0}^I = -\epsilon_{\aleph(i\alpha)I}^{\aleph(i\beta)M}.$$

Combining all of the above, we conclude that if  $\alpha \neq \beta$  then either

$$\tilde{A}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = \tilde{B}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = 0$$

or

$$\tilde{A}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = -\tilde{B}_{\aleph(i\alpha)\aleph(i\beta)}^{MI}.$$

In either case it follows that

$$(96) \quad \tilde{C}_{\aleph(i\alpha)\aleph(i\beta)}^{MI} = 0 \quad \text{whenever } \alpha \neq \beta.$$

Suppose next that  $\alpha = \beta$ ; in this case (90) and (91) become

$$(97) \quad \tilde{A}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = \sum_{L \in \mathcal{I}(N, q+1)} \epsilon_{\aleph(i\alpha)I}^L \cdot \epsilon_{\aleph(i\alpha)M}^L,$$

$$(98) \quad \tilde{B}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = \sum_{K \in \mathcal{I}(N, q-1)} \epsilon_{\aleph(i\alpha)K}^M \cdot \epsilon_{\aleph(i\alpha)K}^I$$

---

<sup>1</sup>If  $q = 0$  or  $q = N-1$  then (92) is equivalent to (93) in the sense that each is false.

and since  $\alpha$ ,  $I$  and  $M$  are fixed, each of these summations consists of at most one term, that is

$$\tilde{A}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = \epsilon_{\aleph(i\alpha)I}^{L_0} \cdot \epsilon_{\aleph(i\alpha)M}^{L_0}; \quad \tilde{B}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = \epsilon_{\aleph(i\alpha)K_0}^M \cdot \epsilon_{\aleph(i\alpha)K_0}^I$$

for at most one choice for each of  $L_0 \in \mathcal{I}(N, q+1)$  and  $K_0 \in \mathcal{I}(N, q-1)$ . In particular we see that

$$(99) \quad I \neq M \Rightarrow \tilde{A}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = \tilde{B}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = 0.$$

On the other hand, for  $I = M$  we have

$$\tilde{A}_{\aleph(i\alpha)\aleph(i\alpha)}^{MM} = (\epsilon_{\aleph(i\alpha)M}^{L_0})^2; \quad \tilde{B}_{\aleph(i\alpha)\aleph(i\alpha)}^{MM} = (\epsilon_{\aleph(i\alpha)K_0}^M)^2$$

for at most one choice of  $L_0$  and of  $K_0$ . We now further distinguish between  $\aleph(i\alpha) \in \{I\}$  and  $\aleph(i\alpha) \notin \{I\}$ . If  $\aleph(i\alpha) \in \{I\}$  then we have  $\tilde{A}_{\aleph(i\alpha)\aleph(i\alpha)}^{MM} = 0$  (because the  $L$ 's do not have repeated terms) and  $\tilde{B}_{\aleph(i\alpha)\aleph(i\alpha)}^{MM} = 1$ . If, instead,  $\aleph(i\alpha) \notin \{I\}$  then we find by the same token that  $\tilde{A}_{\aleph(i\alpha)\aleph(i\alpha)}^{MM} = 1$  and  $\tilde{B}_{\aleph(i\alpha)\aleph(i\alpha)}^{MM} = 0$ . All together this gives

$$(100) \quad \tilde{C}_{\aleph(i\alpha)\aleph(i\alpha)}^{MI} = \begin{cases} 0 & \text{for } M \neq I \\ 1 & \text{for } M = I. \end{cases}$$

Combining (96) and (100) we obtain (67). The proofs of (69) – (71) are obtained in a similar fashion; in this case (49) grants

$$\mathcal{T}_{\ell, \aleph}^* h = (-1)^{k+n\ell} \sum_{\substack{V \in \mathcal{I}(n, q-\ell) \\ I \in \mathcal{I}(n, q) \\ \beta \in (\pi \circ \aleph^{-1})(\mathcal{I}(n, \ell))}} \epsilon_I^{\aleph(i\beta)V} \frac{\partial^k h_I}{\partial x^\beta} dx^V.$$

□

*Proof of Remark 3.1.* If  $\alpha \neq \beta$  and the three conditions in (92) hold, then it follows at once that the first two conditions in (93) are true; by the first condition in (92) we have  $\{I\} = \{I\} \cap \{\aleph(i\alpha)\}^c$ ; combining this identity with the third condition in (92) we obtain  $\{I\} = (\{\aleph(i\beta)\} \cup \{M\}) \cap \{\aleph(i\alpha)\}^c$ , and since  $\alpha \neq \beta$  then  $\{\aleph(i\beta)\} \cap \{\aleph(i\alpha)\}^c = \{\aleph(i\beta)\}$ , and it follows that

$$\{I\} = \{\aleph(i\beta)\} \cup \{Q_0\}, \quad \{Q_0\} := \{M\} \cap \{\aleph(i\alpha)\}^c$$

By the second condition in (92) we have  $\{\aleph(i\beta)\} \cap \{Q_0\} = \emptyset$  and so

$$\{I\} \setminus \{\aleph(i\beta)\} = \{Q_0\}$$

On the other hand, since we have proved that  $\aleph(i\alpha) \in \{M\}$  is true, then we have

$$\{M\} = (\{M\} \cap \{\aleph(i\alpha)\}) \cup (\{M\} \cap \{\aleph(i\alpha)\}^c) = \{\aleph(i\alpha)\} \cup \{Q_0\}$$

and obviously  $\{\aleph(i\alpha)\} \cap \{Q_0\} = \emptyset$ , so

$$\{M\} \setminus \{\aleph(i\alpha)\} = \{Q_0\}$$

which shows that the third condition in (93) is true, as well.

Suppose, conversely, that  $\alpha \neq \beta$  and that the three conditions in (93) hold. Then the first condition in (93) grants  $\{M\} = \{\aleph(i\alpha)\} \dot{\cup} \{P_0\}$  (where  $\dot{\cup}$  denotes disjoint union); similarly, the second condition in (93) grants  $\{I\} = \{\aleph(i\beta)\} \dot{\cup} \{S_0\}$ , and it follows from the third condition in (93) that  $S_0 = P_0$ . Note that in particular  $\aleph(i\alpha) \notin \{P_0\}$  and  $\aleph(i\beta) \notin \{P_0\}$ ; since  $\alpha \neq \beta$ , it follows that the first two conditions in (92) hold. But these (and the above) considerations in turn imply

$$\{\aleph(i\alpha)\} \cup \{I\} = \{\aleph(i\alpha)\} \cup \{\aleph(i\beta)\} \cup P_0 = \{\aleph(i\beta)\} \cup \{M\}$$

which shows that the third condition in (92) is true, as well.  $\square$

*Proof of Lemma 2.7.* The proof is easily reduced to the classical theory via Corollary 2.6 along with (35) and the coordinate-based representations for  $\|\cdot\|_{\tilde{L}^n}$ , see (79). See [CZ], [SR, pg. 62], [S, VI.5] and [T, 13.6].  $\square$

*Theorem 2.3 ( $\ell = 1$ )  $\Rightarrow$  Theorem 2.8.* Without loss of generality we may assume:  $F \in \tilde{C}_{q+1}^{\infty,c}(\mathbb{R}^N)$ ;  $G \in \tilde{C}_{q-1}^{\infty,c}(\mathbb{R}^N)$ , so that  $Z \in \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$ . Write

$$Z = X + Y$$

where

$$(101) \quad \begin{cases} \tilde{T}_{1,\aleph} X &= F \\ \tilde{T}_{1,\aleph}^* X &= 0 \end{cases}$$

and

$$(102) \quad \begin{cases} \tilde{T}_{1,\aleph} Y &= 0 \\ \tilde{T}_{1,\aleph}^* Y &= G \end{cases}$$

We claim that

$$(103) \quad \|X\|_{\tilde{W}_q^{k-1,r}} \leq C \|F\|_{\tilde{L}_{q+1}^1}, \quad r := n/(n-1)$$

and

$$(104) \quad \|Y\|_{\tilde{W}_q^{k-1,r}} \leq C \|G\|_{\tilde{L}_{q-1}^1}, \quad r := n/(n-1)$$

Note that if  $Y$  solves (102) then  $X := *_N Y$  solves (101) with  $F := *_N G$ , and so it suffices to prove (103) for  $F$  and  $X$  as in (101). By duality, this is equivalent to proving

$$(105) \quad |\langle D^{i\beta} X, \varphi \rangle_{\tilde{L}}| \leq C \|F\|_{\tilde{L}_{q+1}^1} \|\varphi\|_{\tilde{L}_q^n}$$

for any  $\beta \in \mathcal{S}(n, s)$  with  $0 \leq s \leq k-1$ , and for any  $\varphi \in \tilde{C}_q^{\infty, c}(\mathbb{R}^N)$ . Let  $\Phi \in \tilde{C}_q^{\infty, c}(\mathbb{R}^N)$  be as in Lemma 2.7. Note that

$$\tilde{T}_{1, \mathbb{N}} D^{i\beta} X = D^{i\beta} \tilde{T}_{1, \mathbb{N}} X = D^{i\beta} F; \quad \tilde{T}_{1, \mathbb{N}}^* D^{i\beta} X = D^{i\beta} \tilde{T}_{1, \mathbb{N}}^* X = 0.$$

By (52) and the above considerations it follows that

$$\langle D^{i\beta} X, \varphi \rangle_{\tilde{L}} = \langle D^{i\beta} F, \tilde{T}_{1, \mathbb{N}} \Phi \rangle_{\tilde{L}} = \langle F, D^{i\beta} \tilde{T}_{1, \mathbb{N}} \Phi \rangle_{\tilde{L}}$$

By Theorem 2.3 ( $\ell = 1$ ;  $H := D^{i\beta} \tilde{T}_{1, \mathbb{N}} \Phi \in \tilde{C}_{q+1}^{\infty, c}(\mathbb{R}^N)$ ) we have

$$|\langle D^{i\beta} X, \varphi \rangle_{\tilde{L}}| \leq C \|F\|_{\tilde{L}_{q+1}^1} \|\nabla D^{i\beta} \tilde{T}_{1, \mathbb{N}} \Phi\|_{\tilde{L}_{q+1}^n} \leq C \|F\|_{\tilde{L}_{q+1}^1} \|\Phi\|_{\widetilde{W}_q^{2k, n}},$$

and it follows from Lemma 2.7 (with  $p := n$ ) that

$$|\langle D^{i\beta} X, \varphi \rangle_{\tilde{L}}| \leq C \|F\|_{\tilde{L}_{q+1}^1} \|\varphi\|_{\tilde{L}_q^n}$$

as desired.  $\square$

*Theorem 2.8  $\Rightarrow$  Theorem 2.3* ( $\ell = 1$ ;  $1 \leq q, p \leq N-1$ ).

We show that (56) holds for any  $q$  in the range  $1 \leq q \leq N-1$ . Suppose that  $\tilde{T}_{1, \mathbb{N}} F = 0$ ,  $F \in \tilde{L}_q^1(\mathbb{R}^N)$  and let  $H \in (\tilde{L}_q^\infty \cap \widetilde{W}_q^{1, n})(\mathbb{R}^N)$ . Without loss of generality we may assume:  $H, F \in \tilde{C}_q^{\infty, c}(\mathbb{R}^N)$ . Let  $X \in \tilde{C}_{q-1}^{\infty, c}(\mathbb{R}^N)$  be the solution of (101) with data  $F$ . Then, by Hölder inequality (42) we have

$$|\langle F, H \rangle_{\tilde{L}}| = |\langle X, \tilde{T}_{1, \mathbb{N}}^* H \rangle_{\tilde{L}}| \lesssim \|X\|_{\widetilde{W}_q^{k-1, r}} \|\tilde{T}_{1, \mathbb{N}}^* H\|_{\widetilde{W}_q^{-(k-1), n}}$$

and it follows from the latter and Theorem 2.8 that

$$|\langle F, H \rangle_{\tilde{L}}| \lesssim \|F\|_{\tilde{L}_q^1} \|\tilde{T}_{1, \mathbb{N}}^* H\|_{\widetilde{W}_{q-1}^{-(k-1), n}}$$

Now observe that if integrate the expression

$$\langle \tilde{T}_{1, \mathbb{N}}^* H, \zeta \rangle_{\tilde{L}}$$

by parts  $(k-1)$ -times and then apply Hölder inequality we obtain

$$|\langle \tilde{T}_{1, \mathbb{N}}^* H, \zeta \rangle_{\tilde{L}}| \leq \|\nabla H\|_{\tilde{L}_q^n} \|\zeta\|_{\widetilde{W}_{q-1}^{k-1, r}}$$

and this leads to the conclusion of the proof of (56) as

$$\|\tilde{T}_{1, \mathbb{N}}^* H\|_{\widetilde{W}_{q-1}^{-(k-1), n}} = \sup_{\|\zeta\|_{\widetilde{W}_{q-1}^{k-1, r}} \leq 1} |\langle \tilde{T}_{1, \mathbb{N}}^* H, \zeta \rangle_{\tilde{L}}|$$

$\square$

## 4. CONCLUDING REMARKS

1. The proof of Theorems 2.3 and 2.8 rely on the specific choice of the embedding  $i : \mathbb{R}^n \rightarrow \mathbb{R}^N$  only to the extent that (20), in fact (21), and (34) hold. This suggests that Theorems 2.3, 2.4 and 2.8 should hold in the more general context of an isometrically embedded manifold

$$\mathcal{M}^{(n)} \hookrightarrow \mathbb{R}^N$$

2. If  $q \geq N - \ell + 1$  or  $p \leq \ell - 1$  then one of the two compatibility conditions (55) and (57) holds trivially and in this case the conclusion of Theorems 2.3 and 2.8 are easily seen to be false: if  $k = 1$  and  $\tilde{T}_{1,\mathbb{N}_0} = d$  (exterior derivative) substitute inequalities hold provided the “defective” data belongs to a suitable (proper) subspace of  $L^1$ , namely the real Hardy space  $H^1(\mathbb{R}^n)$ , see [LS]. We do not know whether substitute inequalities hold when  $k \geq 2$ .
3. In the context of [VS4] our results say the following:
  - $\tilde{T}_{1,\mathbb{N}}$  is canceling from  $V := \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$  to  $E := \tilde{C}_{q+1}^{\infty,c}(\mathbb{R}^N)$  for any  $0 \leq q \leq N - 2$ , see [VS4, Theorem 1.3].
  - $\tilde{T}_{1,\mathbb{N}}^*$  is canceling from  $V := \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$  to  $E := \tilde{C}_{q-1}^{\infty,c}(\mathbb{R}^N)$  for any  $2 \leq q \leq N$ , see [VS4, Theorem 1.3].
  - For any admissible degree increment  $\ell$  and for any  $0 \leq q \leq N - \ell$ ,  $\tilde{T}_{\ell,\mathbb{N}}$  is cocanceling from  $V := \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$  to  $E := \tilde{C}_{q+\ell}^{\infty,c}(\mathbb{R}^N)$ , see [VS4, Propositions 2.1 and 2.2].
  - For any admissible degree increment  $\ell$  and for any  $\ell \leq q \leq N$ ,  $\tilde{T}_{\ell,\mathbb{N}}^*$  is cocanceling from  $V := \tilde{C}_q^{\infty,c}(\mathbb{R}^N)$  to  $E := \tilde{C}_{q-\ell}^{\infty,c}(\mathbb{R}^N)$ , see [VS4, Propositions 2.1 and 2.2].
  - The class  $\mathcal{T}_{\ell,\mathbb{N}}$  has similar properties with  $V = C_q^\infty(\mathbb{R}^n)$  and  $E = C_{q\pm\ell}^\infty(\mathbb{R}^n)$ .

In particular,  $\tilde{T}_{1,\mathbb{N}}$  and  $\mathcal{T}_{1,\mathbb{N}}$  as well as their adjoints, are new examples of canceling operators of *arbitrary* order  $k$ .

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